

N95-25803

~~PRECEDING PAGE BLANK NOT FILMED~~35 *Constrained Mechanics and Variational Problems*A BRIEF SURVEY OF CONSTRAINED MECHANICS AND
VARIATIONAL PROBLEMS IN TERMS OF DIFFERENTIAL FORMS44080
p-15

Robert Hermann

Ever since my graduate student days, I have been impressed and influenced by the elegance and systematization of Mechanics and Variational Calculus contained in Elie Cartan's book "Lecons sur les Invariants Integraux". In the period 1959-69, I expended considerable effort in the development of Cartan's point of view in many books and articles. In this paper (which will appear as a Chapter in "Interdisciplinary Mathematics", v. 30), I will give a quick development of some of the material in my books "Differential Geometry and the Calculus of Variations" and "Geometry, Physics and Systems".

Another purpose in developing this geometric form of the Equations of Mechanics in this Volume is that it fits in with my strategy of investigating mechanics with 'singular' features, such as Delta Functions, Discontinuities, Shocks, etc. As I will show in Volume 30 the C-O-R constructions of Generalized Functions enable one to define 'differential forms with generalized coefficients', thus preparing the ground for the material in this Chapter serving as foundation for Mechanics with Singular Data, the Theory of Splines on nonlinear manifolds, etc. Further, when combined with the Computational Methods under development at the AI Lab of MIT by Gerry Sussman and co-workers this material will be useful in the development of Air Traffic Control methodology.

Another goal of my work is to develop a general structure for ODE systems, to be used in both 'smooth' and 'generalized' (in the sense of Colombeau, Oberguggenberger and Rosinger) Mechanics, Control and Numerical Analysis. Since Martin, Crouch have shown that, in the linear case, Splines may be constructed from linear control system so attention will, in the future, focus on the Splines associated with Generalized Inputs to Nonlinear Control Systems. Work of Sastry and Montgomery indicates that important examples of such systems will be the Left-Invariant Control Systems on Lie Groups, which have been much studied in recent years by

researchers interested in Integrable Systems, Robotics, and Aircraft Guidance,

1. Introduction.

There has been considerable interest recently in constrained mechanics and variational problems. This is in part due to Applied interests (such as 'non-holonomic mechanics in robotics') and in other part due to the fact that several schools of 'pure' mathematics have found that this classical subject is of importance for what they are trying to do. I have made various attempts [2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 27] at developing these subjects since my Lincoln lab days of the late 1950's. In this Chapter, I will sketch a Unified point of view, using Cartan's approach with differential forms. This has the advantage from the C-O-R viewpoint being developed in this Volume that the extension from 'smooth' to 'generalized' data is very systematic and algebraic. (I will only deal with the 'smooth' point of view in this Chapter; I will develop the 'generalized function' material at a later point.) The material presented briefly here about Variational Calculus and Constrained Mechanics can be found in more detail in my books, "Differential Geometry and the Calculus of Variations" "Lie Algebras and Quantum Mechanics", and "Geometry, Physics and Systems".

Here is the basic set-up. Suppose given the following data:

$$\text{A smooth paracompact manifold } X \quad (1.1)$$

$$T(X) \text{ is its tangent vector bundle} \quad (1.2)$$

$$\text{A set } \{\theta, \omega^1, \dots, \omega^m\} \text{ of smooth 1-forms on } X. \quad (1.3)$$

θ is called the action form, $\{\omega^1, \dots, \omega^m\}$ are the constraint forms.

Let us suppose given a curve in X :

37 *Constrained Mechanics and Variational Problems*

$$x = \{t \rightarrow x(t) \in X: a \leq t \leq b\}: [a, b] \rightarrow X \quad (1.4)$$

$$dx/dt = v = \{t \rightarrow dx/dt(t) \in T(X): a \leq t \leq b\}: [a, b] \rightarrow T(X) \quad (1.5)$$

is its **tangent vector** or **velocity curve**.

(In this Chapter, I suppose all such curves are also smooth.)

Definition. The following real number associated to the curve 1.3 is called the **action**:

$$\alpha(x) = \int_{[a, b]} \theta([dx/dt](t)) dt \quad (1.6)$$

The following field of 1-covectors along the curve 1.4 is called the **force**:

$$\{t \rightarrow [dx/dt](t)] d\theta\} \quad (1.7)$$

1.6 and 1.7 are the basic data for both 'mechanics' and 'variational calculus'.

Now, let us deal with 'constraints':

Definition. The curve 1.4 satisfies the constraints associated with the 1-forms $\{\omega^1, \dots, \omega^m\}$ iff. it satisfies the following set of Pfaffian differential equations:

$$0 = \omega^1(dx/dt) = \dots = \omega^m(dx/dt) \quad (1.8)$$

I will show how the basic Equations of Mechanics can be described very compactly and elegantly in terms of this data.

2. The First Variation formula and the Cauchy characteristic curves of $d\theta$.

38 *Constrained Mechanics and Variational Problems*

Keep the data of Section 1. Let us suppose that only the 'action' form θ is given, without any constraints. Let x be a smooth curve in X , given as in 1.4. Suppose that ' s ', $0 \leq s \leq 1$, is a deformation parameter and that $\{s \rightarrow x_s: [a, b] \rightarrow X\}$ is a smooth one-parameter family of curves in X , reducing to the given curve x at ' $s=0$ '. For $t \in [a, b]$, set:

$$v(t) = \text{tangent vector to the curve } \{s \rightarrow x_s(t)\} \text{ at } 't=0'$$

The field $\{t \rightarrow v(t) \in X_{x(t)}\}$ of tangent vectors is called an **infinitesimal deformation of the curve x** . Then:

$$d(\alpha(x_s))/ds \Big|_{s=0} \quad (2.1)$$

is called the **First Variation of the action function** 1.6 along the curve x pointing in the direction of the vector field v . Using the formula 1.6 for the Action, the Cartan Family Identity: ' $V(\theta) = V \rfloor d\theta + d(V \rfloor \theta)$ ': between a differential form and a vector field, and Integration-By-Parts, we have the **First Variation Formula**:

$$d(\alpha(x_s))/ds \Big|_{s=0} = \int_{[a, b]} -[dx/dt] \rfloor d\theta (v(t)) dt + \theta(v)(b) - \theta(v)(a) \quad (2.2)$$

Remark. This formula is a variant of a General Principle:

$$\text{The Variational Derivative of the Action is the Force} \quad (2.3)$$

It also suggests the following:

Definition. A curve $\{x: t \rightarrow x(t)\}$ is called a **Cauchy characteristic curve** for the 2-form $d\theta$ iff:

$$[dx/dt] \rfloor d\theta = 0. \quad (2.4)$$

If x satisfies 2.4, then the First Variation 2.2 vanishes for any infinitesimal deformation v which satisfies the following conditions:

39 *Constrained Mechanics and Variational Problems*

$$\theta(v)(b) = \theta(v)(a) = 0 \quad (2.5)$$

Conditions 2.5 are called **Transversality Conditions**.

At this point, 'symplectic structures and foliations', 'Hamilton's and Lagrange's Equations' (for special choices of θ), etc. enter in a very natural way. See [2, 4, 6, 8, 11, 20, 27, 29].

3. The differential equations of constrained extrema and the augmented action form.

Let us now suppose that $\{\theta, \omega^1, \dots, \omega^m\}$ is given, as in 1.3. Introduce Lagrange Multiplier Variables:

$$\{\lambda_1, \dots, \lambda_m\} \quad (3.1)$$

Consider them as Cartesian coordinates of a copy of R^m . On $X \times R^m$, introduce the following **augmented 1-form**:

$$\theta_{\text{aug}} = \theta + \lambda_1 \omega^1 + \dots + \lambda_m \omega^m \quad (3.2)$$

Then,

$$d\theta_{\text{aug}} = d\theta + d\lambda_1 \wedge \omega^1 + \dots + d\lambda_m \wedge \omega^m + \lambda_1 d\omega^1 + \dots + \lambda_m d\omega^m \quad (3.3)$$

Definition. A curve $\{t \rightarrow x(t)\}$ in X is an **extremal** of the constrained variational problem associated with the differential form data 1.3 if and only if there is a curve in $X \times R^m$ of the form $\{t \rightarrow (x(t), \lambda_1(t), \dots, \lambda_m(t))\}$ which is a Cauchy characteristic curve of $d\theta_{\text{aug}}$. In other words, the 'extremas' are the images under the Cartesian projection map: $\{X \times R^m \rightarrow X\}$ of the Cauchy characteristic curves of $d\theta_{\text{aug}}$.

Theorem 3.1. A curve $\{t \rightarrow (x(t), \lambda_1(t), \dots, \lambda_m(t))\}$ is a Cauchy characteristic curve for the 2-form ' $d\theta_{\text{aug}}$ ' if and only if the following conditions are satisfied:

$$0 = \omega^1(dx/dt) = \dots = \omega^m(dx/dt) \quad (3.4)$$

40 *Constrained Mechanics and Variational Problems*

$$\begin{aligned} dx/dt \rfloor d\theta = & -\lambda_1(t)[dx/dt \rfloor d\omega^1 - \dots - \lambda_m(t)[dx/dt \rfloor d\omega^m \\ & - [d\lambda_1(t)/dt] \omega^1 - \dots - [d\lambda_m(t)/dt] \omega^m \end{aligned} \quad (3.5)$$

Proof. Let v be a tangent vector to the manifold $X \times \mathbb{R}^m$. Then:

$$\begin{aligned} v \rfloor d\theta_{\text{aug}} &= v \rfloor \left(d\theta + d\lambda_1 \wedge \omega^1 + \dots + d\lambda_m \wedge \omega^m + \lambda_1 d\omega^1 + \dots + \lambda_m d\omega^m \right) \\ &= v \rfloor d\theta + v(\lambda_1)\omega^1 + \dots + v(\lambda_m)\omega^m - \omega^1(v)d\lambda_1 - \dots - \omega^m(v)d\lambda_m \\ &\quad + \lambda_1[v \rfloor d\omega^1] + \dots + \lambda_m[v \rfloor d\omega^m] \end{aligned} \quad (3.6)$$

3.6 involves one-forms on $X \times \mathbb{R}^m$. Notice that the only terms on the right hand side of 3.6 which involves $\{d\lambda_1, \dots, d\lambda_m\}$ are the terms $-\omega^1(v)d\lambda_1 - \dots - \omega^m(v)d\lambda_m$. If the tangent vector v is to be Cauchy characteristic these forms must vanish. This leads to the condition 3.4. The conditions 3.5 now follow from inserting 3.4 into the Cauchy characteristic conditions ' $v \rfloor d\theta_{\text{aug}} = 0$ ' and using 3.6.

q.e.d.

Remark. This result expands the treatment to the 'constrained' case that Cartan gave for the 'unconstrained' variational problem in "Lecons sur les Invariants Integraux". See [2] for the connection with the traditional 'Lagrange Variational Problem' as expounded in Caratheodory's book and for the definition and properties of 'symplectic foliations' and further detail.

4. The differential equations of constrained mechanics.

There is considerable confusion in the Literature between the Lagrange Variational Problem (or 'constrained extrema') and 'constrained' (and 'non-holonomic') mechanics'. I will now describe the latter. Suppose again given the following data:

$$\text{A smooth paracompact manifold } X \quad (4.1)$$

41 *Constrained Mechanics and Variational Problems*

$T(X)$ is its tangent vector bundle (4.2)

A set $\{\theta, \omega^1, \dots, \omega^m\}$ of smooth 1-forms on X . (4.3)

Definition. Let $\{x: t \rightarrow x(t)\}$ be a curve in X . It is said to be a trajectory of the constrained mechanical system associated with the data 4.1-4.3 iff. the following conditions are satisfied:

$$0 = \omega^1(dx/dt) = \dots = \omega^m(dx/dt) \quad (4.4)$$

There is a curve in $X \times \mathbb{R}^m$ of the form $\{t \rightarrow (x(t), \mu_1(t), \dots, \mu_m(t))\}$ such that:

$$[dx/dt] \rfloor d\theta = \mu_1(t)\omega^1 + \dots + \mu_m(t)\omega^m \quad (4.5)$$

In other words, 4.4-4.5 define an ODE system whose solutions are curves in $X \times \mathbb{R}^m$. The 'constrained mechanics trajectories' are the projections in X under the Cartesian map projection $\{X \times \mathbb{R}^m \rightarrow X\}$ of the solution curves of the ODE system 4.4-4.5.

5. 'Holonomic' constraints. Equivalence of the Constrained Extremal and Mechanics equations in the 'holonomic' case.

Suppose given the following data:

A smooth paracompact manifold X (5.1)

A set $\{\theta, \omega^1, \dots, \omega^m\}$ of smooth 1-forms on X . (5.2)

Indices $1 \leq a, b, \dots \leq m$ (5.3)

Definition. The constraint forms $\{\omega^a\}$ are said to be holonomic iff. there is a matrix $\{\omega^a_b\}$ of 1-forms such that:

$$d\omega^a = \sum_b \omega^a_b \wedge \omega^b \quad (5.4)$$

42 *Constrained Mechanics and Variational Problems*

Remark. Locally, condition 5.4 is equivalent to the following, more 'geometric', condition:

The Pfaffian System $\{\omega^a = 0\}$ is Frobenius Integrable (5.5)

Let us now combine conditions 5.4 and the Constrained Extremal equations 3.5. The following equations result:

$$dx/dt] d\theta = - \sum_{ab} \lambda_a(t) [dx/dt]] (\omega^a_b \wedge \omega^b) - \sum_a [d\lambda_a(t)/dt] \omega^a \quad (5.6)$$

Rewrite this as follows:

$$\begin{aligned} dx/dt] d\theta = & - \sum_{ab} \lambda_a(t) [(\omega^a_b(dx/dt) \omega^b) - \omega^b(dx/dt) \omega^a_b \\ & - \sum_a [d\lambda_a(t)/dt] \omega^a \end{aligned} \quad (5.7)$$

The second term on the right hand side of 5.7 vanishes as a consequence of the Constraint Equations 4.4, resulting in the following:

$$[dx/dt]] d\theta = - \sum_{ab} \lambda_a(t) [(\omega^a_b(dx/dt) \omega^b) - \sum_a [d\lambda_a(t)/dt] \omega^a \quad (5.7)$$

Theorem 5.1. Let 5.4 be satisfied and let the curve $\{t \rightarrow x(t)\}$ be a solution of the Constrained Extremal Equations. Then, $\{t \rightarrow x(t)\}$ is also a solution of the Constrained Mechanics Equations 4.4-4.5.

Proof. That functions $\{t \rightarrow \mu^a(t)\}$ exist satisfying 4.5 is evident from 5.7.
q.e.d.

Here is the converse:

Theorem 5.2. Let 5.4 be satisfied and let the curve $\{t \rightarrow x(t)\}$ be a solution of the Constrained Mechanics Equations 4.4-4.5. Then, $\{t \rightarrow x(t)\}$ is also a solution of the Constrained Extremal Equations.

43 *Constrained Mechanics and Variational Problems*

Proof. We must show that the existence of functions $\{t \rightarrow \lambda^a(t)\}$ satisfying 5.7 is a consequence of the existence of functions $\{t \rightarrow \mu^a(t)\}$ satisfying 4.4-4.5. Examining the right hand side of 5.7, we see that the $\{\mu^a(t)\}$ can be obtained by solving an ODE whose coefficients depend on the $\{\lambda^a(t)\}$.

q.e.d.

6. The constrained mechanics equations in a 'Hamiltonian' form.

So far, we have been working in the context of general manifold theory. Let us specialize now to the situation which is close to the 'Hamiltonian' formalism in the traditional particle mechanics case.

Suppose given the following data:

$$n \text{ is an integer} \tag{6.1}$$

$$\begin{aligned} \text{The following range of indices:} \\ 1 \leq i, j, \dots \leq n \end{aligned} \tag{6.2}$$

$$X = \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \tag{6.3}$$

$$\{q^i, p_i, t\} \text{ are Cartesian coordinates on } X. \tag{6.4}$$

$$\{(q, p, t) \rightarrow H(q, p, t)\} \text{ is a smooth real-valued function on } X, \text{ called the } \mathbf{Hamiltonian}. \tag{6.5}$$

$$\theta = \sum_i p_i dq^i - H dt \tag{6.6}$$

$$dH = \sum_i H_i dq^i + \sum_i H^i dp_i + H_t dt, \tag{6.7}$$

where $\{H_i, H^i, H_t\}$ are the partial derivatives of the Hamiltonian function with respect to the 'canonical' coordinates 6.4.

44 *Constrained Mechanics and Variational Problems*

$$\text{Theorem 6.1. } d\theta = \sum_i (dq^i - H^i dt) \wedge (dp_i + H_i dt) \quad (6.8)$$

Proof. Follows from 6.7 and 6.6, by a direct computation, which is left to the reader.

Theorem 6.2. Let V be a smooth vector field on X . Then:

$$V \rfloor d\theta = \sum_i (V(q^i) - H^i V(t))(dp_i + H_i dt) - \sum_i (V(p_i) + H_i V(t))(dq^i - H^i dt) \quad (6.9)$$

In particular, if:

$$V(t) = 1 \quad (6.10)$$

then:

$$V \rfloor d\theta = \sum_i (V(q^i) - H^i)(dp_i + H_i dt) - \sum_i (V(p_i) + H_i)(dq^i - H^i dt) \quad (6.11)$$

Proof. Apply the operation ' $V \rfloor$ ' to both sides of 6.8. 6.11 follows from substituting 6.10 into 6.9.

Theorem 6.3. Keep the hypotheses of Theorem 6.2 and condition 6.10. Suppose further that:

$$V \rfloor d\theta = \mu(t) \sum_i a_i dq^i \quad (6.12)$$

where $\{a_i\}$ are smooth functions on X and $\{t \mapsto \mu(t)\}$ is a real-valued function of ' t '. Then, the following relations must be satisfied:

$$V(q^i) = H^i \quad (6.13)$$

$$V(p_i) + H_i = \mu(t) a_i \quad (6.14)$$

$$\sum_i (V(p_i) + H_i) H^i = 0 \quad (6.15)$$

45 *Constrained Mechanics and Variational Problems*

Proof. 6.13-.15 results from combining 6.11 and 6.12, and comparing coefficients of independent coordinate differentials on both sides of the resulting differential form relation.

Theorem 6.4. Let V be the vector field on X defined by 6.10 and 6.12. Then, the orbit curves $\{t \rightarrow (q(t), p(t), t)\}$ of V are solutions of the following ODE's:

$$dq^i/dt = \partial H/\partial p_i \quad (6.16)$$

$$dp_i/dt = -\partial H/\partial q^i + \mu(t)a_i \quad (6.17)$$

$$\sum_i a_i(p(t), q(t), t)[dq^i/dt] = 0 \quad (6.18)$$

Proof. Follows from 6.13-6.15.

Remark. Equations 6.16-6.18 form an ODE system of $(2n+1)$ equations for the $(2n+1)$ 'unknowns': $\{p_i(t), q^i(t), \mu(t)\}$. They are the **Hamiltonian version of the Lagrange Equations of Motion for Constrained Mechanics**. (In this case, there is only one 'constraint, namely 6.18. The case of more constraints can be handled similarly.)

7. Final remarks about generalizations.

The material in Section 6 suggests a Generalization of material about Symplectic Manifolds, Geometric Quantization, etc. from the traditional case abstracted from Particle Mechanics (as in the work of Dirac, van Hove, Segal, Kostant, Souriau, etc) to a abstract situation paralleling the material developed in Section 6.

I will briefly sketch such generalizations. Instead of the ' R^{2n+1} ' situation of Section 6, suppose that we are on a manifold X , with the following relation:

$$d\theta = \Omega - dH \wedge dt \quad (7.1)$$

46 *Constrained Mechanics and Variational Problems*

'H' and 't' are smooth functions on X. θ and Ω are, respectively, a 1- and 2-form on X and Ω is closed. Suppose that V_H is a vector field on X such that:

$$V_H \rfloor d\theta = \mu\omega \quad (7.2)$$

$$V_H(t) = 1, \quad (7.3)$$

where ' ω ' is a 1-form defining the constraints and ' μ ' is a function on X. 7.1-7.3 imply:

$$V_H \rfloor \Omega - V(H)dt + dH = \mu\omega \quad (7.4)$$

This relation generalizes the duality relation between 'infinitesimal symmetries' and 'conserved functions' that plays the basic role in the 'geometric quantization' theory of unconstrained conservative mechanical systems. I plan to study this Geometric Structure further in a later Volume.

Bibliography.

1. R. Hermann, The Differential Geometry of Foliations, II, *J. Math. and Mech.* 11 (1962), pp. 303-316.
2. R. Hermann, Some Differential Geometric Aspects of the Lagrange Variational Problem, *Illinois J. Math.* 6 (1962), pp. 634-673.
3. R. Hermann, On the Accessibility Problem of Control Theory, *Proc. of the Symposium on Differential Equations*, J. Lasalle and S. Lefschetz, Eds., Colorado Springs, Academic Press, 1961.
4. R. Hermann, The Second Variation for Variational Problems in Canonical Form, *Bull. Amer. Math. Soc.* Vol 71 (1965), pp. 145-148.

47 *Constrained Mechanics and Variational Problems*

5. R. Hermann, The Second Variation for Minimal Submanifolds, *J. of Math. and Mech.* 16 (1966), pp. 473-492.
6. R. Hermann, *Differential Geometry and the Calculus of Variations*, Academic Press New York 1969. Second Edition, Math Sci Press, 1977.
7. R. Hermann, Quantum Field Theories with Degenerate Lagrangians, *Phys. Rev.* 177 (1969) P. 2453.
8. R. Hermann, *Lie Algebras and Quantum Mechanics*, W. A. Benjamin New York 1971. 400 pp.
9. R. Hermann, Spectrum-generating Algebras in Classical Mechanics I and II. *J. Math. Phys.* 13. (1972) 873 - 878.
10. R. Hermann, Left Invariant Geodesics and Classical Mechanics on Manifolds, *J. Math. Phys.* 13 (1972) p. 460.
11. R. Hermann, *Geometry Physics and Systems*, Marcel Dekker New York 1973.
12. R. Hermann, (with A. Krener), Non-Linear Controllability and Observability *IEEE Trans. Aut. Cont.* AC-22 (1977), 728-740.
13. R. Hermann, Toda Lattices, Cosymplectic Manifolds, Backlund Transformations and Kinks, Part A and B, Math Sci Press, Brookline, MA 1977. *Interdisciplinary Mathematics*, vols. 15 and 18.
14. R. Hermann, *Quantum and Fermion Differential Geometry*, Part A, Math Sci Press, Brookline, Mass., 1977. *Interdisciplinary Mathematics*, vol. 16.
15. R. Hermann, The Differential geometric Structure of General Mechanical Systems from the Lagrangian Point of View, *J. Math. Phys.* 23 (1982), 2077-2089.

49 *Constrained Mechanics and Variational Problems*

25. R. Hermann, Perturbation Theory for Nonlinear Feedback Control Systems and Goldschmidt-Spencer Integrability of Linear Partial Differential Equations, *Acta App. Math.*, 18, 17-57, 1990
26. R. Hermann, *Geometric Structures in Nonlinear Physics*, Math Sci Press, 1992.
27. R. Hermann, *Constrained Mechanics and Lie Theory*, Math Sci Press, 1993,
28. R. Hermann, *Lie Theoretic Numerical Analysis, Mechanics and Differential Systems*, Math Sci Press, 1994,
29. E. Whittaker, *Analytical Dynamics*, Cambridge University Press.

16. R. Hermann, The Geometric Foundations of the Integrability Property of Differential Equations, I: Lie's "Function Groups"., **J. Math Physics**, 24, 1983, 2422-2432; part II: Mechanics on Affinely Connected Manifolds and the work of Kowalewska and Painleve, **J. Math Physics**, 25m 1984, 778-785.
17. R. Hermann, **Topics in The Geometric Theory of Integrable Systems**, Math Sci Press, 1984.
18. R. Hermann, **Topics in Physical Geometry**, Math Sci Press, 1988
19. R. Hermann, Invariants for Feedback Equivalence and Cauchy Characteristic Multifoliations for Nonlinear Control Systems, **Acta App. Math.**, 11, 1988, 123-153.
20. R. Hermann, Differential Form Methods in the Theory of Variational Systems and Lagrangian Field Theories , **Acta App. Math.**, 12, 1988, 35-78.
21. R. Hermann, Dynamical Systems and Infinite Dimensional Lie Algebras of the "Current Algebra" or "Kac-Moody" Type, in **Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems**, AIP Conference Proceedings vol. 88, Am. Inst. of Physics, 67-72.
22. R. Hermann, Lax Representation as a "Quantization" of the "Function Groups" of Sophus Lie, **Phys. Rev. D**, 26 , 1982, 1491-1492.
23. R. Hermann, Geometric Construction and Properties of Some families of Solutions of Nonlinear Partial Differential Equations, **J. Math. Phys.**, 24, 1983, 510-521.
24. R. Hermann, Nonlinear Feedback Control and Systems of Partial Differential Equations, **Acta App. Math.**, 17, 41-94, 1989.

